STABILITY OF FLOW OF A VISCOUS LIQUID THROUGH A ROTATING RADIAL CHANNEL AT LOW ROSSBY NUMBERS

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UDC 532.516

1. Statement of the Problem. We consider the steady stabilized flow of an incompressible viscous liquid through a prismatic channel with a rectangular cross section, rotating with a constant angular velocity ω about an axis through the center of a cross section of the channel perpendicular to one of its sides.

We introduce a Cartesian coordinate system $0\,x'y'z'$ rigidly attached to the channel with the Oy' axis along the axis of rotation and the Oz' axis along the axis of the channel in the direction of flow. The linear dimension of the cross section of the channel is 2h in the direction of the y' axis and $2l$ in the direction of the x' axis. We assume that the liquid flows in the channel under the action of a constant longitudinal modified pressure gradient $\partial\Pi/\partial z' = \alpha$ at a Rossby number $\mathrm{Ro} = U/\omega L \ll 1$.

Under the above assumptions the motion of the liquid in the channel is described by the following system of differential equations [1]:

$$\Delta\Delta\psi = \mathrm{R}\partial w/\partial y,\ \Delta w = -\mathrm{R}\partial\psi/\partial y - 2, \tag{1.1}$$

where

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2;\ x = x'/L;\ y = y'/L;$$
$$\psi = \psi'/UL;\ w = w'/U;\ \mathrm{R} = 2\omega L^2/\nu;$$
$$\Pi = \frac{p}{\rho} - \frac{\omega^2}{2}(x'^2 + z'^2);\quad L = \begin{cases} l & \text{for}\quad h \geqslant l, \\ h & \text{for}\quad h \leqslant l; \end{cases}$$

$U = -\alpha L^2/2\nu$ is the characteristic velocity, w', z' component of the relative velocity vector; ψ^1, stream function of the transverse flow; p, pressure, ρ, density, and ν, kinematic viscosity of the liquid.

The boundary conditions for system (1.1) have the form

$$w = \psi = \frac{\partial\psi}{\partial x} = \partial\psi/\partial y = 0\quad \text{at}\quad x = \pm\, l/L;\ y = \pm h/L. \tag{1.2}$$

2. A Channel Extending in the Direction of the Axis of Rotation ($h \geq l$). The solution of system (1.1) which satisfies the boundary conditions

$$w(\pm 1,\ y) = 0,\ w(x, \pm\ 1/\varepsilon) = 0,\ \psi(\pm 1,\ y) = 0,$$

$\psi_y'(x,\ \pm\ 1/\varepsilon) = 0$, can be written in the form

$$\psi = \sum_{k=0}^{\infty} A_k \left[\frac{\operatorname{ch}(b_1 x)}{\operatorname{ch} b_1} - r_1 \operatorname{ch}(b_2 x)\cos(b_3 x) + r_2 \operatorname{sh}(b_2 x)\cdot\sin(b_3 x)\right]\times$$
$$\times\sin(\beta_k y) + \sum_{k=0}^{\infty} B_k \left[\frac{\alpha_k^2 - b_k^2}{b_4}\frac{\operatorname{sh}(b_4 y)}{\operatorname{ch} c_4} + r_3 \operatorname{sh}(b_5 y)\cdot\cos(b_6 y) -\right.$$
$$\left. - r_4 \operatorname{ch}(b_5 y)\cdot\sin(b_6 y)\right]\cdot\cos(\alpha_k x) + 4\sum_{k=0}^{\infty}\frac{(-1)^k}{\alpha_k^3 \mathrm{R}}[r_5 \operatorname{sh}(b_5 y)\cdot\cos(b_6 y) -$$
$$- r_6 \operatorname{ch}(b_5 y)\cdot\sin(b_6 y)]\cos(\alpha_k x); \tag{2.1}$$

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 76-83, January-February, 1980. Original article submitted January 17, 1979.

 0021-8944/80/2101-0070$07.50

$$w = 1 - x^2 + 4 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\alpha_k^3} [r_7 \operatorname{ch}(b_5 y) \cdot \cos(b_6 y) - r_8 \operatorname{sh}(b_5 y) \cdot \sin(b_6 y)] \times \tag{2.2}$$

$$\times \cos(\alpha_k x) - \sum_{k=0}^{\infty} A_k \sqrt[3]{\mathrm{R}\beta_k} \left[\frac{\operatorname{ch}(b_1 x)}{\operatorname{ch} b_1} - r_9 \operatorname{ch}(b_2 x) \cdot \cos(b_3 x) - \right.$$

$$\left. - r_{10} \operatorname{sh}(b_2 x) \cdot \sin(b_3 x) \right] \cos(\beta_k y) + \sum_{k=0}^{\infty} \mathrm{R} B_k \left[\frac{\operatorname{ch}(b_4 y)}{\operatorname{ch} c_4} - r_{11} \operatorname{ch}(b_5 y) \cdot \cos(b_6 y) - r_{12} \operatorname{sh}(b_5 y) \cdot \sin(b_6 y) \right] \cos(\alpha_k x),$$

where

$$\alpha_k = \frac{\pi}{2}(2k+1)\varepsilon_1; \quad \beta_k = \frac{\pi}{2}(2k+1)\varepsilon; \tag{2.3}$$

$$\varepsilon_1 = \begin{cases} 1 & \text{for } h \geqslant l \\ h/l & \text{for } h \leqslant l \end{cases}; \quad \varepsilon = \begin{cases} l/h & \text{for } h \geqslant l, \\ 1 & \text{for } h \leqslant l; \end{cases}$$

$$b_1 = \sqrt[3]{\mathrm{R}\beta_k}\sqrt{1+t_k}; \quad t_k = \sqrt[3]{\beta_k^4/\mathrm{R}^2};$$

$$b_{2,3} = \frac{1}{2}\sqrt[3]{\mathrm{R}\beta_k}\sqrt{2\sqrt{1-t_k+t_k^2} \pm (2t_k - 1)}; \quad b_4 = \sqrt{\alpha_k^2 + v_2 - v_1};$$

$$b_{5,6} = \frac{1}{2}\sqrt{\sqrt{(2\alpha_k^2 + v_1 - v_2)^2 + 3(v_1 + v_2)^2} \pm (2\alpha_k + v_1 - v_2)};$$

$$v_{1,2} = \sqrt[3]{\sqrt{\frac{(\mathrm{R}\alpha_k)^4}{4} + \frac{\mathrm{R}^6}{27}} \pm \frac{(\mathrm{R}\alpha_k)^2}{2}};$$

$$r_1 = F(\sqrt{3}\operatorname{sh} b_2 \cdot \sin b_3 + \operatorname{ch} b_2 \cdot \cos b_3);$$

$$r_2 = F(\sqrt{3}\operatorname{ch} b_2 \cdot \cos b_3 - \operatorname{sh} b_2 \cdot \sin b_3);$$

$$r_3 = pF_1(b_6 \operatorname{ch} c_5 \cdot \cos c_6 + b_5 \operatorname{sh} c_5 \cdot \sin c_6) + gF_1[b_5(b_5^2 + b_6^2 - $$
$$- \alpha_k^2)\operatorname{sh} c_5 \cdot \sin c_6 - b_6(b_5^2 + b_6^2 + \alpha_k^2)\operatorname{ch} c_5 \cdot \cos c_6];$$

$$r_4 = pF_1(b_5 \operatorname{ch} c_5 \cdot \cos c_6 - b_6 \operatorname{sh} c_5 \cdot \sin c_6) + gF_1[b_5(b_5^2 + b_6^2 - $$
$$- \alpha_k^2)\operatorname{ch} c_5 \cdot \cos c_6 + b_6(b_5^2 + b_6^2 + \alpha_k^2)\operatorname{sh} c_5 \cdot \sin c_6];$$

$$r_5 = pF_1(b_6 \operatorname{ch} c_5 \cdot \cos c_6 + b_5 \operatorname{sh} c_5 \cdot \sin c_6);$$

$$r_6 = pF_1(b_5 \operatorname{ch} c_5 \cos c_6 - b_6 \operatorname{sh} c_5 \cdot \sin c_6);$$

$$r_7 = F_1(\theta \operatorname{ch} c_5 \cdot \cos c_6 + \operatorname{sh} c_5 \cdot \sin c_6);$$

$$r_8 = F_1(\theta \operatorname{ch} c_5 \cdot \cos c_6 - \operatorname{sh} c_5 \cdot \sin c_6);$$

$$r_9 = F(\operatorname{ch} b_2 \cdot \cos b_3 - \sqrt{3}\operatorname{sh} b_2 \cdot \sin b_3);$$

$$r_{10} = F(\sqrt{3}\operatorname{ch} b_2 \cdot \cos b_3 + \operatorname{sh} b_2 \cdot \sin b_3);$$

$$r_{11} = F_1(\operatorname{ch} c_5 \cdot \cos c_6 + \theta_1 \operatorname{sh} c_5 \cdot \sin c_6);$$

$$r_{12} = F_1(\operatorname{sh} c_5 \cdot \sin c_6 - \theta_1 \operatorname{ch} c_5 \cdot \cos c_6);$$

$$F = \frac{1}{\operatorname{ch}^2 b_2 - \sin^2 b_3}; \quad F_1 = \frac{1}{\operatorname{ch}^2 b_5 - \sin^2 b_6};$$

$$c_i = b_i/\varepsilon \ (i = 4,\ 5,\ 6);$$

$$\theta = \frac{b_5^2 - b_6^2 - \alpha_k^2}{2b_5 b_6}; \quad \theta_1 = \frac{b_5^2 - b_6^2 - b_4^2}{2b_5 b_6};$$

$$p = \frac{(b_5^2 - b_6^2 - \alpha_k^2)^2 + 4b_5^2 b_6^2}{2b_5 b_6(b_5^2 + b_6^2)}; \quad g = \frac{\alpha_k^2 - b_4^2}{2b_5 b_6(b_5^2 + b_6^2)}.$$

The coefficients A_k and B_k in (2.1) and (2.2) must be determined from the remaining boundary conditions for the stream function ψ. If A_k and B_k are determined from the conditions

$$\psi\left(x, \pm \frac{1}{\varepsilon}\right) = 0, \quad \psi'_x = (\pm 1, y) = 0, \tag{2.4}$$

it is easy to show that all the boundary conditions of the problem will be satisfied. Because of the symmetry of conditions (1.2) and the fact that w and ψ are even functions, it is sufficient to satisfy Eqs. (2.4) only for $y = 1/\varepsilon$ and $x = 1$, respectively. From the first of Eqs. (2.4) it follows that

$$\sum_{k=0}^{\infty} (-1)^{k+1} A_k H_k(x) = \sum_{k=0}^{\infty} \left(B_k Q_k + \frac{4(-1)^k}{\alpha_k^3 R} T_k\right) \cos(\alpha_k x), \tag{2.5}$$

where

$$Q_k = r_3 \operatorname{sh} c_5 \cdot \cos c_6 - r_4 \operatorname{ch} c_5 \cdot \sin c_6 + \frac{\alpha_k^2 - b_k^2}{b_4} \operatorname{th} c_4;$$

$$T_k = r_5 \operatorname{sh} c_5 \cdot \cos c_6 - r_6 \operatorname{ch} c_5 \cdot \sin c_6;$$

$$H_k = \frac{\operatorname{ch}(b_1 x)}{\operatorname{ch} b_1} - r_1 \operatorname{ch}(b_2 x) \cdot \cos(b_3 x) + r_2 \operatorname{sh}(b_2 x) \sin(b_3 x).$$

We expand the function $H_k(x)$ in a cosine series

$$H_k(x) = \sum_{i=0}^{\infty} \Omega_{ik} \cos(\alpha_i x) \quad (k = 0, 1, 2 \ldots),$$

$$\alpha_i = \frac{\pi}{2}(2i + 1)\varepsilon_1, \quad |x| \leqslant 1,$$

$$\Omega_{ik} = 2\alpha_i (-1)^i \left\{\frac{1}{b_1^2 + \alpha_i^2} - \frac{b_2^2 - b_3^2 + \alpha_i^2 + 2\sqrt{3}\, b_2 b_3}{(b_2^2 + b_3^2 + \alpha_i^2)^2 - 4b_3^2\alpha_i^2}\right\}.$$

We substitute the expansion found for $H_k(x)$ into the left-hand side of (2.5) and interchange the order of summations, replacing subscripts k by j and i by k. In addition, by equating coefficients of identical cosine terms we obtain the following infinite system of linear equations:

$$B_k = \frac{4(-1)^{k+1}}{\alpha_k^3 R} \frac{T_k}{Q_k} - \sum_{j=0}^{\infty} (-1)^j \Omega_{kj} \frac{A_j}{Q_k} \quad (k = 0, 1, 2 \ldots). \tag{2.6}$$

Using the second of Eqs. (2.4), we have

$$\sum_{k=0}^{\infty} A_k \Gamma_k \sin(\beta_k y) = \sum_{k=0}^{\infty} (-1)^k \alpha_k B_k \Phi_k(y) + \frac{4}{R} \sum_{k=0}^{\infty} \frac{Z_k(y)}{\alpha_k^2}, \tag{2.7}$$

where

$$\Gamma_k = b_1 \operatorname{th} b_1 + (r_2 b_3 - r_1 b_2) \operatorname{sh} b_2 \cdot \cos b_3 + (r_2 b_2 + r_3 b_3) \operatorname{ch} b_2 \cdot \sin b_3;$$

$$Z_k(y) = r_5 \operatorname{sh}(b_5 y) \cdot \cos(b_6 y) - r_6 \operatorname{ch}(b_5 y) \cdot \sin(b_6 y);$$

$$\Phi_k(y) = \frac{\alpha_k^2 - b_k^2}{b_4} \frac{\operatorname{sh}(b_4 y)}{\operatorname{ch} c_4} + r_3 \operatorname{sh}(b_5 y) \cdot \cos(b_6 y) - r_4 \operatorname{ch}(b_5 y) \cdot \sin(b_6 y).$$

We expand the functions $Z_k(y)$ and $\Phi_k(y)$ in sine series

$$Z_k(y) = \sum_{i=0}^{\infty} \tau_{ik} \sin(\beta_i y), \quad \Phi_k(y) = \sum_{i=0}^{\infty} \varphi_{ik} \sin(\beta_i y),$$

$$\beta_i = \frac{\pi}{2}(2i + 1)\varepsilon, \quad |y| \leqslant \frac{1}{\varepsilon} \quad (k = 0, 1, 2 \ldots),$$

$$\tau_{ik} = \frac{2\varepsilon(-1)^i\left[\left(b_5^2 - b_6^2 - \alpha_k^2\right)^2 + 4b_5^2b_6^2\right]}{\left(b_5^2 + b_6^2 + \beta_i^2\right)^2 + 4b_5^2b_6^2},$$

$$\varphi_{ik} = 2\varepsilon(-1)^i\left[\frac{\alpha_k^2 - b_4^2}{b_4^2 + \beta_i^2} + \frac{\left(b_5^2 - b_6^2 - \alpha_k^2\right)^2 + 4b_5^2b_6^2 - \left(\alpha_k^2 + \beta_i^2\right)\left(\alpha_k^2 - b_4^2\right)}{\left(b_5^2 + b_6^2 + \beta_i^2\right)^2 - 4b_5^2b_6^2}\right].$$

We substitute the expansions for $Z_k(y)$ and $\Phi_k(y)$ into the right-hand side of Eqs. (2.7) and interchange the order of summations, replacing subscripts k by j and i by k.

In addition, by equating coefficients of identical sine terms we obtain the infinite system of linear equations

$$A_k = \sum_{j=0}^{\infty}(-1)^j \alpha_j \varphi_{kj}\frac{B_j}{T_k} + \frac{4}{RT_k}\sum_{j=0}^{\infty}\frac{\tau_{kj}}{\alpha_j^2} \quad (k = 0, 1, 2 \ldots). \tag{2.8}$$

The coefficients A_k and B_k must be determined from (2.6) and (2.8). It can be shown that for specific values of the parameters R and ε this system has a unique bounded solution which can be found, for example, by the method of successive approximations. If this is done, the problem posed is completely solved.

The average velocity of the liquid along the channel is given by the relation

$$w_0 = \frac{w_0'}{U} = \frac{1}{hl}\int_0^h\int_0^l w(x', y')\,dx'dy'. \tag{2.9}$$

Substituting into this the value of w from (2.1), we obtain

$$w_0 = \frac{2}{3} - f(\varepsilon, R); \tag{2.10}$$

$$f(\varepsilon, R) = 2\varepsilon\sum_{k=0}^{\infty} F_1\frac{(b_5 + \theta b_6)\,\mathrm{sh}\,(2c_5) + (b_6 - \theta b_5)\sin(2c_6)}{\alpha_k^4\left(b_5^2 + b_6^2\right)} - \tag{2.11}$$

$$- \varepsilon\sqrt[3]{R}\sum_{k=0}^{\infty}\frac{(-1)^k A_k}{\beta_k^{2/3}}\left[\frac{\mathrm{th}\,b_1}{b_1} - F\frac{\left(b_2 - \sqrt{3}b_3\right)\mathrm{sh}\,(2b_2) + \left(b_3 + \sqrt{3}b_2\right)\sin(2b_3)}{2\left(b_2^2 + b_3^2\right)}\right] -$$

$$- 2\varepsilon\sum_{k=0}^{\infty}\frac{(-1)^k B_k}{\alpha_k}\left[\frac{\mathrm{th}\,b_4}{b_4} - F_1\frac{(b_5 + \theta_1 b_6)\,\mathrm{sh}\,(2c_5) + (b_6 - \theta_1 b_5)\sin(2c_6)}{2\left(b_5^2 + b_6^2\right)}\right].$$

The resistance coefficient for the flow of liquid through the rotating channel is

$$\lambda_\omega = -\frac{4\alpha L}{(w_0')^2}. \tag{2.12}$$

Substituting into this the values of α and w_0', we have

$$\lambda_\omega = \frac{16}{\mathrm{Re}\left[\frac{2}{3} - f(\varepsilon, R)\right]}, \quad \mathrm{Re} = \frac{2w_0'l}{\nu}. \tag{2.13}$$

The resistance coefficient of a stationary channel λ_0 is a special case of (2.13) for R = 0. Taking this into account, the ratio of the resistance coefficients of rotating and stationary channels is

$$\frac{\lambda_\omega}{\lambda_0} = \frac{\frac{2}{3} - f(\varepsilon, 0)}{\frac{2}{3} - f(\varepsilon, R)}. \tag{2.14}$$

The expression for f (ε, 0) is obtained from (2.11) by going to the limit R → 0:

$$f(\varepsilon, 0) = 4\varepsilon \sum_{k=0}^{\infty} \frac{\operatorname{th}(\alpha_k/\varepsilon)}{\alpha_k^5}.$$

If ε goes to zero in (2.14), $\lambda\omega/\lambda_0 \to 1$, i.e., the resistance of a rapidly rotating slot channel extending along the axis of rotation is close to the resistance of a stationary channel.

3. A Channel Extending in a Direction Perpendicular to the Axis of Rotation ($l \geqslant h$). Using the familiar expansion

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{2k+1}{2}\pi y\right) = \begin{cases} \pi/4 & \text{for } |y| < 1, \\ 0 & \text{for } |y| = 1 \end{cases} \tag{3.1}$$

system (1.1) can be written in the form

$$\Delta\Delta\psi = \mathrm{R}\frac{\partial w}{\partial y}, \tag{3.2}$$

$$\Delta w = -\mathrm{R}\frac{\partial \psi}{\partial y} - \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos\left(\frac{2k+1}{2}\pi y\right).$$

Calculations similar to those in Sec. 2 give the following solution of system (3.2) which satisfies the boundary conditions (1.2):

$$\psi = 2\mathrm{R}\sum_{k=0}^{\infty}\frac{(-1)^k}{(\mathrm{R}^2+\beta_k^4)\beta_k^2}[s_1 \operatorname{ch}(b_2x)\cdot\cos(b_3x) + s_2 \operatorname{sh}(b_2x)\cdot\sin(b_2x) - 1]\times \tag{3.3}$$

$$\times\sin(\beta_k y) + \frac{8}{\sqrt{3}}\frac{1}{\sqrt[3]{\mathrm{R}}}\sum_{k=0}^{\infty}\frac{(-1)^k\beta_k^{2/3}}{\mathrm{R}^2+\beta_k^4}[s_3\operatorname{ch}(b_2x)\cdot\cos(b_3x) -$$

$$- s_4\operatorname{sh}(b_2x)\cdot\sin(b_3x)]\sin(\beta_k y) + \sum_{k=0}^{\infty}D_k\left[\frac{\operatorname{ch}(b_1x)}{\operatorname{ch}c_1} - s_5\operatorname{ch}(b_2x)\cdot\cos(b_3x) + \right.$$

$$\left. + s_6\operatorname{sh}(b_2x)\cdot\sin(b_3x)\right]\sin(\beta_k y) + \sum_{k=0}^{\infty}E_k\left[\frac{\alpha_k^2 - t_k^2}{b_4}\frac{\operatorname{sh}(b_4y)}{\operatorname{ch}b_4} + \right.$$

$$\left. + s_7\operatorname{sh}(b_5y)\cdot\cos(b_6y) - s_8\operatorname{ch}(b_5y)\cdot\sin(b_6y)\right]\cos(\alpha_k x);$$

$$w = -\sum_{k=0}^{\infty}\sqrt[3]{\mathrm{R}\beta_k}D_k\left[\frac{\operatorname{ch}(b_1x)}{\operatorname{ch}c_1} - s_9\operatorname{ch}(b_2x)\cos(b_3x) - \right. \tag{3.4}$$

$$\left. - s_{10}\operatorname{sh}(b_2x)\cdot\sin(b_3x)\right]\cos(\beta_k y) + 8\frac{\mathrm{R}^{4/3}}{\sqrt{3}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(R^2+\beta_k^4)\beta_k^{5/3}}\times$$

$$\times[s_{11}\operatorname{ch}(b_2x)\cdot\cos(b_3x) - s_{12}\operatorname{sh}(b_2x)\cdot\sin(b_3x)]\cos(\beta_k y) +$$

$$+ 4\sum_{k=0}^{\infty}\frac{(-1)^k\beta_k}{\mathrm{R}^2+\beta_k^4}[1 + s_{13}\operatorname{ch}(b_2x)\cdot\cos(b_3x) - s_{14}\operatorname{sh}(b_2x)\cdot\sin(b_3x)]\cos(\beta_k y) +$$

$$+ \mathrm{R}\sum_{k=0}^{\infty}E_k\left[\frac{\operatorname{ch}(b_4y)}{\operatorname{ch}b_4} - s_{15}\operatorname{ch}(b_5y)\cdot\cos(b_6y) - s_{16}\operatorname{sh}(b_5y)\cdot\sin(b_6y)\right]\cos(\alpha_k y),$$

where

$$s_1 = F_2\left(\operatorname{ch}c_2\cdot\cos c_3 + \frac{1}{\sqrt{3}}\operatorname{sh}c_2\cdot\sin c_3\right);$$

$$s_2 = F_2\left(\operatorname{sh} c_2 \cdot \sin c_3 - \frac{1}{\sqrt{3}} \operatorname{ch} c_2 \cdot \cos c_3\right);$$

$$s_3 = F_3 \operatorname{sh} c_2 \cdot \sin c_3; \quad s_4 = F_3 \operatorname{ch} c_2 \cdot \cos c_3;$$

$$s_5 = F_2 (\operatorname{ch} c_2 \cdot \cos c_3 + \sqrt{3} \operatorname{sh} c_2 \cdot \sin c_3);$$

$$s_6 = F_2 (\sqrt{3} \operatorname{ch} c_2 \cdot \cos c_3 - \operatorname{sh} c_2 \cdot \sin c_3);$$

$$s_7 = pF_3 (b_6 \operatorname{ch} b_5 \cdot \cos b_6 + b_5 \operatorname{sh} b_5 \cdot \sin b_6) + gF_2 [b_5 (b_5^2 + b_6^2 - \alpha_k^2) \times \times \operatorname{sh} b_5 \cdot \sin b_6 - b_6 (b_5^2 + b_6^2 + \alpha_k^2) \operatorname{ch} b_5 \cdot \cos b_6];$$

$$s_8 = pF_3 (b_5 \operatorname{ch} b_5 \cdot \cos b_6 - b_6 \operatorname{sh} b_5 \cdot \sin b_6) + gF_2 [b_5 (b_5^2 + b_6^2 - \alpha_k^2) \times \times \operatorname{ch} b_5 \cdot \cos b_6 + b_6 (b_5^2 + b_6^2 + \alpha_k^2) \operatorname{sh} b_5 \cdot \sin b_6];$$

$$s_9 = F_2(\operatorname{ch} c_2 \cdot \cos c_3 - \sqrt{3} \operatorname{sh} c_2 \cdot \sin c_3);$$

$$s_{10} = F_2(\sqrt{3} \operatorname{ch} c_2 \cdot \cos c_3 + \operatorname{sh} c_2 \cdot \sin c_3);$$

$$s_{11} = F_2 \operatorname{sh} c_2 \cdot \sin c_3; \quad s_{12} = F_2 \operatorname{ch} c_2 \cdot \cos c_3;$$

$$s_{13} = F_2\left(\frac{1}{\sqrt{3}} \operatorname{sh} c_2 \cdot \sin c_3 - \operatorname{ch} c_2 \cdot \cos c_3\right);$$

$$s_{14} = F_2\left(\operatorname{sh} c_2 \cdot \sin c_3 + \frac{1}{\sqrt{3}} \operatorname{ch} c_2 \cdot \cos c_3\right);$$

$$s_{15} = F_3(\operatorname{ch} b_5 \cdot \cos b_6 + \theta_1 \operatorname{sh} b_5 \cdot \sin b_6);$$

$$s_{16} = F_3(\operatorname{sh} b_5 \cdot \sin b_6 - \theta_1 \operatorname{ch} b_5 \cdot \cos b_6);$$

$$F_2 = \frac{1}{\operatorname{ch}^2 c_2 - \sin^2 c_3}; \quad F_3 = \frac{1}{\operatorname{ch}^2 b_5 - \sin^2 b_6};$$

$$c_i = b_i/\varepsilon_1 (i = 1,\ 2,\ 3).$$

In Eqs. (3.3) and (3.4) the coefficients D_k and E_k are roots of the following infinite system of linear equations:

$$E_k = -\frac{4R}{N_k} \sum_{j=0}^{\infty} \frac{1}{R^2 + \beta_k^4} \left[\frac{\sigma_{kj}}{\beta_j^2} + \frac{2}{\sqrt{3}} \left(\frac{\beta_j}{R^2}\right)^{2/3} \eta_{kj}\right] + \frac{1}{N_k} \sum_{j=0}^{\infty} (-1)^{k+1} h_{kj} D_j, \quad (3.5)$$

$$D_k = \frac{4(-1)^{k+1}}{R^2 + \beta_k^4} \left[\frac{R^2}{\beta_k^2} \frac{H_{1k}}{H_{3k}} + \frac{2}{\sqrt{3}} \left(\frac{\beta_k^2}{R}\right)^{1/3} \frac{H_{2k}}{H_{3k}}\right] + \frac{1}{H_{3k}} \sum_{j=0}^{\infty} (-1)^j \alpha_j \rho_{kj} E_j,$$

where

$$N_k = \frac{\alpha_k^2 - b_4^2}{b_4} \operatorname{th} b_4 + s_7 \operatorname{sh} b_5 \cdot \cos b_6 - s_8 \operatorname{ch} b_5 \cdot \sin b_6;$$

$$H_{1k} = \frac{F_2}{2} \left[\left(b_2 - \frac{b_3}{\sqrt{3}}\right) \operatorname{sh}(2c_3) - \left(\frac{b_2}{\sqrt{3}} + b_3\right) \operatorname{sh}(2c_3)\right];$$

$$H_{2k} = -\frac{F_2}{2} [b_2 \sin(2c_3) + b_3 \operatorname{sh}(2c_2)];$$

$$H_{3k} = b_1 \operatorname{th} c_1 + \frac{F_2}{2} [(b_2 \sqrt{3} + b_3) \sin(2c_3) + (\sqrt{3} b_3 - b_2) \operatorname{sh}(2c_2);$$

$$\rho_{kj} = 2(-1)^k \left[\frac{\alpha_j^2 - b_{4j}^2}{\beta_k^2 + b_{4j}^2} + \frac{(b_{5j}^2 - b_{6j}^2 - \alpha_k^2)^2 + 4b_{5j}^2 b_{6j}^2 - (\alpha_j^2 - b_{4j}^2)(\alpha_j^2 + \beta_k^2)}{(b_{5j}^2 + b_{6j}^2 + \beta_k^2)^2 - 4b_{5j}^2 \beta_k^2}\right]; \quad (3.6)$$

$$\sigma_{kj} = 2\varepsilon_1 (-1)^k \left[\frac{\alpha_k \left(b_{2j}^2 - b_{3j}^2 + \alpha_k^2 + \frac{2}{\sqrt{3}} b_{2j} b_{3j}\right)}{(b_{2j}^2 + b_{3j}^2 + \alpha_k^2)^2 - 4b_{3j}^2 \alpha_k^2} - \frac{1}{\alpha_k}\right];$$

$$h_{kj} = 2\varepsilon_1 (-1)^k \alpha_k \left[\frac{1}{b_{1j}^2 + \alpha_k^2} - \frac{b_{2j}^2 - b_{3j}^2 + \alpha_k^2 + 2\sqrt{3}\, b_{2j} b_{3j}}{(b_{2j}^2 + b_{3j}^2 + \alpha_k^2)^2 - 4b_{3j}^2 \alpha_k^2}\right];$$

$$\eta_{kj} = 4\varepsilon_1 (-1)^k \frac{\alpha_k b_{2j} b_{3j}}{(b_{2j}^2 + b_{3j}^2 + \alpha_k^2)^2 - 4b_{3j}^2 \alpha_k^2}.$$

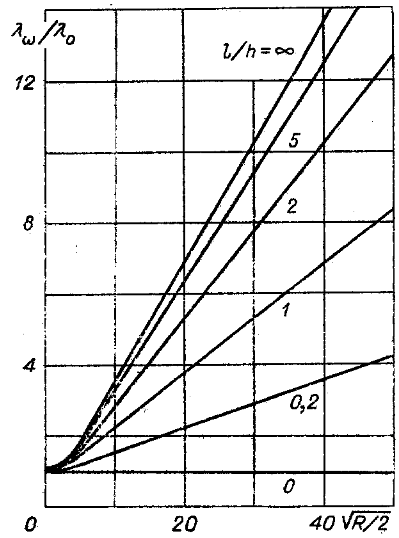

Fig. 1

In (3.6) the extra subscript j on b_i indicates that k must be replaced by j in the corresponding formulas (2.3) for b_i.

Let us calculate the average velocity of the liquid along the channel. Substituting the value of w from (3.4) into (2.9), we have

$$w_0 = f_1(\varepsilon_1, \mathrm{R}) = 4\sum_{k=0}^{\infty} \frac{1}{\mathrm{R}^2+\beta_k^4}\left\{1 - \varepsilon_1 F_3 \times \right. \tag{3.7}$$

$$\left. \times \frac{\left[\left(b_3 + b_2/\sqrt{3}\right)\sin(2b_3) + \left(b_2 - b_3/\sqrt{3}\right)\mathrm{sh}(2b_2)\right]}{2\left(b_2^2+b_3^2\right)}\right\} +$$

$$+ \frac{4\mathrm{R}^{4/3}}{\sqrt{3}} \sum_{k=0}^{\infty} \frac{F_3}{\left(\mathrm{R}^2+\beta_k^4\right)\beta_k^{8/3}} \frac{\left(b_3\,\mathrm{sh}(2b_2) - b_2\sin(2b_3)\right)}{b_2^2+b_3^2} +$$

$$+ \mathrm{R}\sum_{k=0}^{\infty} \frac{(-1)^k}{\beta_k} E_k \left\{\frac{\mathrm{th}\, b_4}{b_4} - F_3 \frac{\left(b_5 + \theta_1 b_6\right)\mathrm{sh}(2b_5) + \left(b_6 - \theta_1 b_5\right)\sin(2b_6)}{2\left(b_5^2+b_6^2\right)}\right\} -$$

$$- \sqrt[3]{\mathrm{R}}\,\varepsilon_1 \sum_{k=0}^{\infty} \frac{D_k(-1)^k}{\beta_k^{2/3}} \left[\frac{\mathrm{th}\, c_1}{b_1} - F_3 \frac{\left(b_3 + \sqrt{3}\, b_2\right)\sin(2b_3) + \left(b_2 - \sqrt{3}\, b_3\right)\mathrm{sh}(2b_2)}{2\left(b_2^2+b_3^2\right)}\right\}.$$

Equations (3.3), (3.7), and (3.4) as $l \to \infty$ ($\varepsilon_1 \to 0$) go over into corresponding expressions for the longitudinal component of velocity and the stream function of secondary flow in a rotating slot channel extending in a direction perpendicular to the axis of rotation [2]; for R = 0 these equations give the velocity distribution in a stationary channel.

Using (1.2), (2.12), and (3.7), we obtain the following expression for the ratio of the friction coefficients of rotating and stationary channels:

$$\frac{\lambda_\omega}{\lambda_0} = \frac{f_1(\varepsilon_1, 0)}{f_1(\varepsilon_1, \mathrm{R})}.$$

The expression for f_1 (ε_1, 0) is obtained from (3.7) by going to the limit R → 0

$$f_1(\varepsilon_1, 0) = \frac{2}{3} - f(\varepsilon_1, 0).$$

Equations (2.6), (2.8), and (3.3) were solved by the iteration method for various values of the parameter R and l/h of practical interest. We note that boundary-value problem (1.1) and (1.2) can be solved directly on existing computers only for relatively small values of R

[3]. The values found for the coefficients A_k, B_k, D_k, and E_k were used to calculate the velocity distribution and the resistance coefficient of the rotating channel. Figure 1 shows the calculated values of the resistance coefficient of the channel λ_ω/λ_0 as a function of the parameter $\sqrt{R/2}$ for various values of l/h. For fixed R the resistance coefficient of the channel increases with an increase in its extension in a direction perpendicular to the axis of rotation. For fixed l/h and small values of R the ratio λ_ω/λ_0 is proportional to R; for R > 300 the dependence of λ_ω/λ_0 on $\sqrt{R/2}$ is practically linear. This shows that for large R the main contribution to the channel resistance comes from the Ekman layer formed on the channel walls perpendicular to the axis of rotation.

The lack of experimental data prevents a direct comparison of our calculated results with experiment. A comparison of the theoretical values of the resistance coefficient with the corresponding values of λ_ω obtained by extrapolating the experimental law $\lambda_\omega = \lambda_\omega$ (Ro) for R = const in the range of small Rossby numbers shows good agreement for a channel with a square cross section for all values of R.

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MOTION OF A PLANE PLATE OF FINITE WIDTH IN A VISCOUS CONDUCTIVE LIQUID, PRODUCED BY ELECTROMAGNETIC FORCES

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UDC 538.4

Studies are available [1-4] which demonstrate the possibility, in principle, of creating magnetohydrodynamic engines for marine vessels. They have demonstrated that due to the low conductivity of seawater and the limited value of the magnetic fields employed, the efficiency of such engines will be low. However, recent successes in development of superconductive materials permit the hope of increased field intensities in such magnetic systems, and consequently, increased efficiencies in such MHD engines. It is thus of interest to study the peculiarities of flow around bodies in the vicinity of which volume electromagnetic forces produced by a source within the body flowed over exist.

1. The present study is dedicated to examination of the motion of the simplest model of a body (a plate of finite width) in a viscous conductive liquid. Numerical solution of the Navier—Stokes equation together with the equation of motion of the solid will determine the velocity of the plate's translational motion relative to the liquid which is at rest at infinity, and also the pattern of flow around the plate; the plate is set in motion by a magnetic field in the form of a traveling wave created by surface currents distributed over the plate width. The presence of turbulent volume forces in the liquid near the plate set in motion in this electromagnetic fashion makes the flow pattern different from the classical one.

Becuase of the numerical method used to solve the Navier—Stokes equation the flow under study must be limited to Reynolds number values on the order of magnitude of 10^3.

Thus, we will consider a plane plate of width $2a$ along the x axis, infinite in extent along the z axis, and located in an infinite viscous conductive liquid. Along the z axis a surface current

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 84-91, January-February, 1980. Original article submitted January 30, 1979.

0021-8944/80/2101-0077$07.50